**INDIAN STATISTICAL INSTITUTE**
**Semestral Exam**
**Introduction to Representation Theory**
**2018-2019**

Total marks: 100
Time: 3 hours

You may assume $G$ to be a finite group and all representations of $G$ to be over $\mathbb{C}$.

1. Give justifications for the following statements.
(a) If $\chi$ is the character of an irreducible representation of $S_n$ not equivalent to the sign representation, then $\sum_{\sigma \in S_n} sgn(\sigma)\chi(\sigma) = 0$.
(b) Let $\chi$ be an irreducible character of $G$. Then for every element $g$ in $Z(G)$, the center of $G$, we have $\chi(g) = \zeta\chi(1)$, where $\zeta$ is a root of unity.
(c) All characters of the symmetric group $S_n$ are real.
(d) Order of a group $G$ is odd if and only if $G$ does not have any non-trivial real irreducible characters. (5× 4)

2. (a) Define *algebraic integers*.
(b) Let $\phi : G \longrightarrow GL_d(\mathbb{C})$ be an irreducible representation.
(i) Let $g \in G$ and let $h$ be the size of the conjugacy class of $g$. Then show that $h\chi_\phi(g)/d$ is an algebraic integer.
(ii) Show that $d$ divides $o(G)$. (4+10+6)

3. (a) Let $\sigma : G \longrightarrow S_X$ be a group action. Show that $dim\ \mathbb{C}X^G$, the fixed subspace of the action of $G$ on $\mathbb{C}X$, is the number of orbits of $G$ on $X$.
(b) Write down the character table of $S_4$, with proper justification. (10+10)

4. (a) Let $G$ be a group and $H$ a subgroup of $G$. Let $\phi : H \longrightarrow GL_d(\mathbb{C})$ be a representation of $H$. Define the induced representation $Ind_H^G\phi$ of $G$.
(b) Let $G$ be the dihedral group $D_{2n} = \{r^m, sr^m | 0 \leq m \leq n-1\}$. Let $H =< r >\subset G$. For $0 \leq k \leq n-1$, let $\chi_k : H \longrightarrow \mathbb{C}^*$ be the representation given by $\chi_k(r^m) = e^{2\pi ikm/n}$. Determine $Ind_H^G\chi_k$. (8+12)

5. Compute the character table of $S_5$. (20)